Error Propagation in Nuclear Reaction Data Measurement

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1 Poisson distribution

It is often assumed in nuclear physics experiments that the probability to observe n events during the measurement follows the Poisson distribution:

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}. \tag{1}$$

Show that

1. the function $P(n)$ is normalized to probability distribution, namely $\sum_{n=0}^{\infty} P(n) = 1$. Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2. the mean value $\langle n \rangle = \sum_{n=0}^{\infty} nP(n)$ is λ.

3. the variance $\mathrm{Var}(n) = \sum_{n=0}^{\infty} n^2 P(n) - \langle n \rangle^2$ is λ.

4. the standard deviation Δn is $\sqrt{\lambda}$.

2 Estimation of irradiation time

One wants to measure the ^{235}U(n,f) cross section at 5 MeV within 1% accuracy by using an uranium-235 layer (areal number density $n = 5 \times 10^{-6}$ atoms/barn) in a fission chamber and a neutron source (flux $\phi = 2 \times 10^5$ neutrons/sec).

1. The number of the fission is $N = \phi n \sigma t$, where t is the irradiation time. It is known that the cross section is about 1 b. How many fission events are expected after 1-hr irradiation?

2. How long we have to irradiate the sample in order to measure the cross section with the uncertainty due to counting statistics of 1%? Assume that (1) the number of observed fission events follows the Poisson distribution, (2) the number of observed fission events represents the mean value of the distribution, (3) the statistical uncertainty is the standard deviation of the distribution. Note that the fractional uncertainty in the cross section $\Delta\sigma$ due to counting statistics is related with the fractional uncertainty in the number of the observed reaction N by $\Delta\sigma/\sigma = \Delta N/N$.

3 Time-of-flight measurement

The time-of-flight (TOF) method is a typical method to determine the kinetic energy of neutrons E by measuring the velocity of neutrons v. If we measure the time-of-flight t with the flight path length L,

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{L}{t}\right)^2 \tag{2}$$

in the non-relativistic approximation.

1. The fractional uncertainty in the energy is related with the fractional uncertainties in the flight path length and time-of-flight by

$$\left(\frac{\Delta E}{E}\right)^2 = s_{EL}^2\left(\frac{\Delta L}{L}\right)^2 + s_{Et}^2\left(\frac{\Delta t}{t}\right)^2 \tag{3}$$

   with the relative sensitivity coefficients

$$\begin{aligned} s_{EL} &= \frac{L}{E}\frac{\partial E}{\partial L}, \\ s_{Et} &= \frac{t}{E}\frac{\partial E}{\partial t}. \end{aligned} \tag{4}$$

$$\tag{5}$$

   Calculate these relative sensitivity coefficients.

2. We would like to measure the energy with an accuracy of 0.1%. We can determine the flight path length with an accuracy of 1 cm whereas the uncertainty in the time-of-flight is negligible. How long flight path should we keep for the measurement?

4 Uncertainty propagation to thermal cross section

The thermal (2200 m/s) neutron capture cross section σ measured by the activation method under a mono-energetic neutron field can be derived by the following data reduction equation:

$$\sigma = \frac{C}{\phi n \epsilon I} \frac{\lambda}{(1 - e^{-\lambda t_i}) e^{-\lambda t_c} (1 - e^{-\lambda t_m})} \tag{6}$$

where C, λ, ϕ, n, ϵ, I are the number of counts, decay constant, neutron flux, number of target atoms per area, detection efficiency and gamma intensity. Also t_i, t_c and t_m are the irradiation time, cooling time, and measurement time.

1. There are 10 parameters on the right-hand side of the data reduction equation. The uncertainties in some parameters can be propagated to the uncertainty in the cross section by the quadrature sum rule, namely

$$\left(\frac{\Delta y}{y}\right)^2 = \sum_{i=1}^{n} \left(\frac{\Delta x_i}{x_i}\right)^2 . \tag{7}$$

   for $y = y(x_1, x_2, ..., x_n)$. However this rule is not valid for some parameters in the data reduction equation. Identify four such parameters.

2. The quadrature sum rule is generalized to

$$\left(\frac{\Delta y}{y}\right)^2 = \sum_{i=1}^{n} s_i^2 \left(\frac{\Delta x_i}{x_i}\right)^2 , \tag{8}$$

   where $s_i = (x_i/y)\partial y/\partial x_i$ is the relative sensitivity coefficient. Calculate the relative sensitivity coefficients for C, ϕ and λ for error propagation from $\Delta C/C$, $\Delta\phi/\phi$ and $\Delta\lambda/\lambda$ to $\Delta\sigma/\sigma$.

3. When the neutron field is not mono-energetic but thermalized neutrons (i.e., thermally equilibrated at the room temperature) plus epithermal neutrons, one can still determine the thermal neutron capture cross section σ by using a "Cd-filter" which removes neutrons below 0.55 eV from the neutron field. By using the counts without the filter (C) and with the filter (C'), the data reduction equation is modified to

$$\sigma = \frac{C - C'/F}{\phi n x \epsilon I g} \frac{\lambda}{(1 - e^{-\lambda t_i}) e^{-\lambda t_c} (1 - e^{-\lambda t_m})} \tag{9}$$

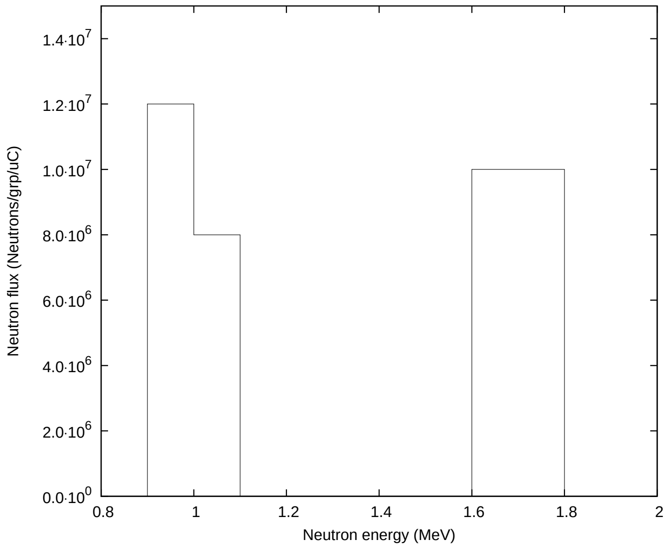
   where F and g are the Cd transmission factor and Westcott factor. Calculate the relative sensitivity coefficient for error propagation from $\Delta C/C$ to $\Delta\sigma/\sigma$ for this extended data reduction equation.

5 Uncertainty propagation to averaged cross section

Use EXCEL for numerical calculation to keep enough number of digits.

There are two neutron fields which group-wise neutron energy spectra Φ_k are summarized with the group-wise evaluated cross sections of a standard reaction σ_k as follows:

Field (i)	Group (k)	$E_{\min}$ MeV	$E_{\max}$ MeV	Φ_k neutrons/grp/μC	σ_k b	$\Delta\sigma_k/\sigma_k$ %	Correlation coefficient			
1	1	0.9	1.0	1.2×10^7	0.800	1	1.00			
	2	1.0	1.1	0.8×10^7	0.700	1	0.10	1.00		
2	3	1.6	1.7	1.0×10^7	0.600	1	0.05	0.05	1.00	
	4	1.7	1.8	1.0×10^7	0.600	1	0.05	0.05	0.10	1.00



1. Calculate the spectrum averaged standard cross sections for the two neutron fields $\langle\sigma\rangle_i = \sum_{k=1}^{n} w_{ik}\sigma_k$, where the weighting factor w_{ik} is $w_{ik} = \Phi_k / \sum_{l=1}^{n} \Phi_l$.

2. Calculate the fractional uncertainties in the averaged standard cross sections $\Delta\langle\sigma\rangle_i \,/\, \langle\sigma\rangle_i$. Use $\left(\frac{\Delta y_i}{y_i}\right)^2 = \sum_{k=1}^{n}\sum_{l=1}^{n} s_{ik}\left(\frac{\Delta x_k}{x_k}\right)\mathrm{Cor}(x_k, x_l)\left(\frac{\Delta x_l}{x_l}\right)s_{il}$ with the relative sensitivity coefficient $s_{ik} = \frac{x_k}{y_i}\left(\frac{\partial y}{\partial x_k}\right)_{y=y_i}$ for $y = y(x_1, x_2, ...x_n)$.

3. Calculate the fractional covariance $\mathrm{cov}(\langle\sigma\rangle_1, \langle\sigma\rangle_2)$ and correlation coefficient $\mathrm{Cor}(\langle\sigma\rangle_1, \langle\sigma\rangle_2)$, where the fractional covariance is defined as $\mathrm{cov}(x_1, x_2) = \mathrm{Cov}(x_1, x_2)/(x_1 x_2)$. Note that $\mathrm{cov}(y_i, y_j) = \sum_{k=1}^{n}\sum_{l=1}^{n} s_{ik}\mathrm{cov}(x_k, x_l)s_{jl}$ and $\mathrm{Cor}(y_i, y_j) = \mathrm{cov}(y_i, y_j) \bigg/ \left(\frac{\Delta y_i}{y_i}\frac{\Delta y_j}{y_j}\right)$

6 Uncertainty propagation to interpolated detection efficiency

Use EXCEL for numerical calculation to keep enough number of digits.

One measured the detection efficiencies $\epsilon(E)$ of a germanium detector for several gamma-lines, and parameterized the result by

$$\epsilon(E;\epsilon_0,\epsilon_c,E_0) = \epsilon_0 \exp(-E/E_0) + \epsilon_c. \tag{10}$$

1. Calculate three partial derivatives $S_0(E) = \partial\epsilon(E)/\partial\epsilon_0$, $S_E(E) = \partial\epsilon(E)/\partial E_0$ and $S_c(E) = \partial\epsilon(E)/\partial\epsilon_c$.

2. When p dependence of a quantity y is parameterized by n parameters $x_i(i = 1, n)$ such as $y(p) = y(p; x_1, x_2, ..., x_n)$, the covariance between $y(p)$ and $y(q)$ is propagated from the covariance of $\{x_i\}$ by

$$\mathrm{Cov}(y(p), y(q)) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial y}{\partial x_i}\right)_{y=y(p)} \mathrm{Cov}(x_i, x_j)\left(\frac{\partial y}{\partial x_j}\right)_{y=y(q)} \tag{11}$$

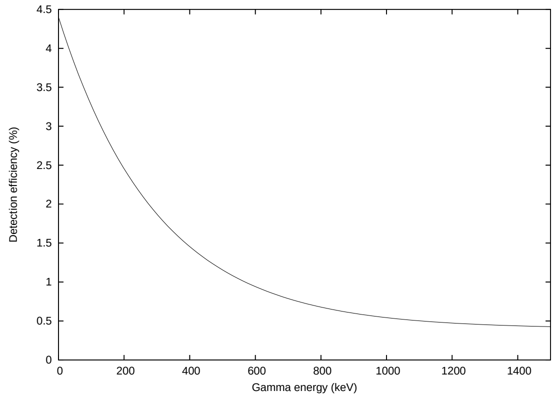
   Show that the covariance of the detection efficiencies at two energies E_i and E_j is

$$\begin{aligned}\mathrm{Cov}\left(\epsilon(E_i),\epsilon(E_j)\right) &= e^{-\frac{E_i+E_j}{E_0}}(\Delta\epsilon_0)^2 + \frac{\epsilon_0^2 E_i E_j}{E_0^4} e^{-\frac{E_i+E_j}{E_0}}(\Delta E_0)^2 + (\Delta\epsilon_c)^2 \\ &+ \epsilon_0 \frac{E_i+E_j}{E_0^2} e^{-\frac{E_i+E_j}{E_0}} \mathrm{Cov}(\epsilon_0, E_0) \\ &+ \left(e^{-\frac{E_i}{E_0}} + e^{-\frac{E_j}{E_0}}\right)\mathrm{Cov}(\epsilon_0,\epsilon_c) \\ &+ \frac{\epsilon_0}{E_0^2}\left(E_i e^{-\frac{E_i}{E_0}} + E_j e^{-\frac{E_j}{E_0}}\right)\mathrm{Cov}(E_0,\epsilon_c). \end{aligned} \tag{12}$$

   Note that $(\Delta x_i)^2 = \mathrm{Var}(x_i) = \mathrm{Cov}(x_i, x_i)$.

3. The result of the fitting is summarized as follows:

Parameter	Value	Uncertainty	Correlation coefficient		
ϵ_0	4.0	0.1	1.0		
E_0 (keV)	300	10	-0.8	1.0	
ϵ_c	0.40	0.01	0.4	-0.7	1.0



Calculate the three variances $(\Delta\epsilon_0)^2$, $(\Delta E_0)^2$ and $(\Delta\epsilon_c)^2$ as well as three covariances $\mathrm{Cov}(\epsilon_0, E_0)$, $\mathrm{Cov}(\epsilon_0, \epsilon_c)$ and $\mathrm{Cov}(E_0, \epsilon_c)$. Note that $\mathrm{Cov}(x, y) = \mathrm{Cor}(x, y)\Delta x\Delta y$, where $\mathrm{Cor}(x, y)$ is the correlation coefficient between x and y.

4. Calculate the interpolated detection efficiencies and their uncertainties at 800 keV (ϵ_{800}) and 1000 keV (ϵ_{1000}) as well as the covariance between them. Use $(\Delta\epsilon(E))^2 = \mathrm{Var}\,(\epsilon(E)) = \mathrm{Cov}\,(\epsilon(E), \epsilon(E))$ for calculation of the uncertainties.

5. Calculate the fractional uncertainties $\Delta\epsilon_{800}/\epsilon_{800}$ and $\Delta\epsilon_{1000}/\epsilon_{1000}$, and fractional covariance

$$\mathrm{cov}\,(\epsilon_{800}, \epsilon_{1000}) = \mathrm{Cov}\,(\epsilon_{800}, \epsilon_{1000})\,/\,(\epsilon_{800} \cdot \epsilon_{1000})\,.$$

6. Show that the fractional variance of the efficiency ratio $\eta(E_i, E_j) = \epsilon(E_i)/\epsilon(E_j)$ is

$$\mathrm{var}\left(\eta(E_i, E_j)\right) = \mathrm{var}\,(\epsilon(E_i)) + \mathrm{var}\left(\epsilon(E_j)\right) - 2\,\mathrm{cov}\left(\epsilon(E_i), \epsilon(E_j)\right), \tag{13}$$

where the fractional variance and fractional covariance are defined as $\mathrm{var}(x_i) = \mathrm{Var}(x_i)/x_i^2$ and $\mathrm{cov}(x_i, x_j) = \mathrm{Cov}(x_i, x_j)/(x_i \cdot x_j)$, respectively. Use

$$\mathrm{var}(y) = \sum_{i=1}^{n}\sum_{j=1}^{n} s_i \mathrm{cov}(x_i, x_j) s_j$$

with

$$s_i = \frac{x_i}{y}\frac{\partial y}{\partial x_i}$$

for $y = y(x_1, x_2, ..., x_n)$.

7. Calculate the efficiency ratio $\eta_{800,1000} = \epsilon_{800}/\epsilon_{1000}$ and its fractional uncertainty.

7 Answers

1-1
$$\sum_{n=0}^{\infty} P(n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} e^{\lambda} = 1$$

1-2
$$\langle n \rangle = \sum_{n=0}^{\infty} nP(n) = \sum_{n=1}^{\infty} nP(n) = \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} = \lambda \sum_{n=1}^{\infty} P(n-1) = \lambda \sum_{n=0}^{\infty} P(n) = \lambda$$

1-3
$$\begin{aligned}
\sum_{n=0}^{\infty} n^2 P(n) &= \sum_{n=1}^{\infty} n^2 P(n) = \lambda \sum_{n=1}^{\infty} n \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} = \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} + \lambda \sum_{n=1}^{\infty} (n-1) \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} \\
&= \lambda \sum_{n=1}^{\infty} P(n-1) + \lambda \sum_{n=1}^{\infty} (n-1) P(n-1) = \lambda \sum_{n=0}^{\infty} P(n) + \lambda \sum_{n=0}^{\infty} nP(n) = \lambda + \lambda^2.
\end{aligned}$$

Therefore
$$\mathrm{Var}(n) = \sum_{n=0}^{\infty} n^2 P(n) - \langle n \rangle^2 = \sum_{n=0}^{\infty} n^2 P(n) - \lambda^2 = \lambda.$$

1-4
$$\Delta n = \sqrt{\mathrm{Var}(n)} = \sqrt{\lambda}$$

2-1
$$N = \phi n \sigma t = 2 \times 10^5 \cdot 5 \times 10^{-6} \cdot 1 \cdot 3.6 \times 10^3 = 3.6 \times 10^3.$$

2-2 From the answer of the first question, the number of the reactions expected during t-hr irradiation is $N(t) = 3.6t \times 10^3$ and its statistical uncertainty is $\Delta N(t) = \sqrt{N(t)} = \sqrt{3.6t \times 10^3} = 60\sqrt{t}$. In order to make the fractional statistical uncertainty $\Delta N(t)/N(t) = 1/\sqrt{N(t)}$ to 1%, we have to irradiate the foil for $t = (1/60/0.01)^2 \sim 2.8$ hr.

3-1 $s_{EL} = 2$, $s_{Et} = -2$.

3-2 By using the answer to the first question,
$$\left(\frac{\Delta E}{E}\right)^2 = 4\left(\frac{\Delta L}{L}\right)^2.$$
when the uncertainty in the time-of-flight is negligible. This equation shows that we have to maintain the uncertainty in the flight path length within 0.05% to maintain the fractional uncertainty in E within 0.1%. If we can measure the flight path length with 1 cm accuracy (i.e., ΔL=1 cm), then we need the flight path length of 1 cm/0.05=20 cm.

4-1 λ, t_i, t_c and t_m because they are related to σ through exponential functions.

4-2
$$\begin{aligned}
s_C &= \frac{C}{\sigma}\frac{\partial \sigma}{\partial C} = 1, \\
s_\phi &= \frac{\phi}{\sigma}\frac{\partial \sigma}{\partial \phi} = -1
\end{aligned}$$

are obvious because σ is proportional or inversely proportional to these two parameters.

In order to calculate s_λ, we define the four variables: $A = C/(\phi n \epsilon I)$, $\Lambda_1 = 1 - e^{-\lambda t_i}$, $\Lambda_2 = e^{-\lambda t_c}$ and $\Lambda_3 = 1 - e^{-\lambda t_m}$. Then the data reduction equation is

$$\sigma = A \frac{\lambda}{\Lambda_1 \Lambda_2 \Lambda_3}.$$

Then

$$\frac{\partial \sigma}{\partial \lambda} = \frac{A}{(\Lambda_1 \Lambda_2 \Lambda_3)^2} \left(\Lambda_1 \Lambda_2 \Lambda_3 - \lambda \frac{\partial \Lambda_1 \Lambda_2 \Lambda_3}{\partial \lambda} \right).$$

The partial derivative in the second term is

$$\begin{aligned} \frac{\partial \Lambda_1 \Lambda_2 \Lambda_3}{\partial \lambda} &= \frac{\partial \Lambda_1}{\partial \lambda} \Lambda_2 \Lambda_3 + \frac{\partial \Lambda_2}{\partial \lambda} \Lambda_1 \Lambda_3 + \frac{\partial \Lambda_3}{\partial \lambda} \Lambda_2 \Lambda_3 \\ &= \Lambda_1 \Lambda_2 \Lambda_3 \left(\frac{t_i e^{-\lambda t_i}}{\Lambda_1} - \frac{t_c e^{-\lambda t_c}}{\Lambda_2} + \frac{t_m e^{-\lambda t_m}}{\Lambda_3} \right). \end{aligned}$$

Therefore

$$\begin{aligned} s_\lambda &= \frac{\lambda}{\sigma} \frac{\partial \sigma}{\partial \lambda} \\ &= 1 - \lambda \left(\frac{t_i e^{-\lambda t_i}}{\Lambda_1} - \frac{t_c e^{-\lambda t_c}}{\Lambda_2} + \frac{t_m e^{-\lambda t_m}}{\Lambda_3} \right) \\ &= 1 - \frac{\lambda t_i e^{-\lambda t_i}}{1 - e^{-\lambda t_i}} + \lambda t_c - \frac{\lambda t_m e^{-\lambda t_m}}{1 - e^{-\lambda t_m}}. \end{aligned}$$

The second to fourth terms show the deviation from the quadrature sum rule in the error propagation. Note that $s_\lambda \to -1$ when λt_i, λt_c and $\lambda t_m \to 0$.

4-3

$$s_C = \frac{C}{\sigma} \frac{\partial \sigma}{\partial C} = \frac{C}{C - C'/F} = \left(1 - \frac{C'}{CF} \right)^{-1}$$

The second terms show the deviation from the quadrature sum rule in the error propagation.

5-1

$$\begin{aligned} <\sigma>_1 &= w_{11}\sigma_1 + w_{12}\sigma_2 = \frac{\Phi_1}{\Phi_1 + \Phi_2}\sigma_1 + \frac{\Phi_2}{\Phi_1 + \Phi_2}\sigma_2 = 0.760 \text{ b}, \\ <\sigma>_2 &= w_{23}\sigma_3 + w_{24}\sigma_4 = \frac{\Phi_3}{\Phi_3 + \Phi_4}\sigma_3 + \frac{\Phi_4}{\Phi_3 + \Phi_4}\sigma_4 = 0.600 \text{ b}. \end{aligned}$$

5-2

$$\begin{aligned} \left(\frac{\Delta \langle \sigma \rangle_i}{\langle \sigma \rangle_i} \right)^2 &= \sum_k \sum_l \left[\frac{\sigma_k}{\langle \sigma \rangle_i} \frac{\partial \langle \sigma \rangle_i}{\partial \sigma_k} \right] \frac{\Delta \sigma_k}{\sigma_k} \mathrm{Cor}(\sigma_k, \sigma_l) \frac{\Delta \sigma_l}{\sigma_l} \left[\frac{\partial \langle \sigma \rangle_i}{\partial \sigma_l} \frac{\sigma_l}{\langle \sigma \rangle_i} \right] \\ &= \sum_k \sum_l w_{ik} w_{il} \frac{\sigma_k \sigma_l}{\langle \sigma \rangle_i^2} \frac{\Delta \sigma_k}{\sigma_k} \frac{\Delta \sigma_l}{\sigma_l} \mathrm{Cor}(\sigma_k, \sigma_l), \end{aligned}$$

where the k and l under the summation symbols run from 1 to 2 for the field 1, and from 3 to 4 for the field 2. Namely

$$\begin{aligned}
\left(\frac{\Delta\langle\sigma\rangle_1}{\langle\sigma\rangle_1}\right)^2 &= w_{11}^2\frac{\sigma_1^2}{\langle\sigma\rangle_1^2}\left(\frac{\Delta\sigma_1}{\sigma_1}\right)^2 + w_{12}^2\frac{\sigma_2^2}{\langle\sigma\rangle_1^2}\left(\frac{\Delta\sigma_2}{\sigma_2}\right)^2 + w_{11}w_{12}\frac{\sigma_1\sigma_2}{\langle\sigma\rangle_1^2}\left(\frac{\Delta\sigma_1}{\sigma_1}\right)\left(\frac{\Delta\sigma_2}{\sigma_2}\right)\mathrm{Cor}(\sigma_1,\sigma_2)\\
&\sim 0.558\%^2,\\
\left(\frac{\Delta\langle\sigma\rangle_2}{\langle\sigma\rangle_2}\right)^2 &= w_{23}^2\frac{\sigma_3^2}{\langle\sigma\rangle_2^2}\left(\frac{\Delta\sigma_3}{\sigma_3}\right)^2 + w_{24}^2\frac{\sigma_4^2}{\langle\sigma\rangle_2^2}\left(\frac{\Delta\sigma_4}{\sigma_4}\right)^2 + w_{23}w_{24}\frac{\sigma_3\sigma_4}{\langle\sigma\rangle_2^2}\left(\frac{\Delta\sigma_3}{\sigma_3}\right)\left(\frac{\Delta\sigma_4}{\sigma_4}\right)\mathrm{Cor}(\sigma_3,\sigma_4)\\
&\sim 0.525\%^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\Delta\langle\sigma\rangle_1}{\langle\sigma\rangle_1}\right) &\sim 0.747\%,\\
\left(\frac{\Delta\langle\sigma\rangle_2}{\langle\sigma\rangle_2}\right)^2 &\sim 0.725\%.
\end{aligned}$$

These results show that the fractional uncertainties of the averaged cross sections (~0.7%) are smaller than those in the original group-wise cross sections (1%).

5-3

$$\begin{aligned}
\mathrm{cov}(\langle\sigma\rangle_1,\langle\sigma\rangle_2) &= \sum_{k=1}^{2}\sum_{l=3}^{4}\frac{\sigma_k}{\langle\sigma\rangle_1}\frac{\partial\langle\sigma\rangle_1}{\partial\sigma_k}\mathrm{cov}(\sigma_k,\sigma_l)\frac{\partial\langle\sigma\rangle_2}{\partial\sigma_l}\frac{\sigma_l}{\langle\sigma\rangle_2}\\
&= \sum_{k=1}^{2}\sum_{l=3}^{4} w_{1k}w_{2l}\frac{\sigma_k}{\langle\sigma\rangle_1}\frac{\sigma_l}{\langle\sigma\rangle_2}\mathrm{cov}(\sigma_k,\sigma_l)\\
&= w_{11}w_{23}\frac{\sigma_1}{\langle\sigma\rangle_1}\frac{\sigma_3}{\langle\sigma\rangle_2}\mathrm{cov}(\sigma_1,\sigma_3) + w_{11}w_{24}\frac{\sigma_1}{\langle\sigma\rangle_1}\frac{\sigma_4}{\langle\sigma\rangle_2}\mathrm{cov}(\sigma_1,\sigma_4)\\
&+ w_{12}w_{23}\frac{\sigma_2}{\langle\sigma\rangle_1}\frac{\sigma_3}{\langle\sigma\rangle_2}\mathrm{cov}(\sigma_2,\sigma_3) + w_{12}w_{24}\frac{\sigma_2}{\langle\sigma\rangle_1}\frac{\sigma_4}{\langle\sigma\rangle_2}\mathrm{cov}(\sigma_2,\sigma_4) = 0.05\%^2\\
\mathrm{Cor}(\langle\sigma\rangle_1,\langle\sigma\rangle_2) &= \mathrm{cov}(\langle\sigma\rangle_1,\langle\sigma\rangle_2)\Bigg/\left(\frac{\Delta\langle\sigma\rangle_1}{\langle\sigma\rangle_1}\frac{\Delta\langle\sigma\rangle_2}{\langle\sigma\rangle_2}\right) \sim 0.05/0.75/0.72 \sim 0.09.
\end{aligned}$$

We can summarize the averaged cross sections, their uncertainties and correlation as follows:

Field (i)	$E_{\min}$ MeV	$E_{\max}$ MeV	$\langle\sigma\rangle_i$ b	$\Delta\langle\sigma\rangle_i\,/\,\langle\sigma\rangle_i$ %	Correlation coefficient	
1	0.9	1.1	0.76	0.75	1.00	
2	1.6	1.8	0.60	0.72	0.09	1.00

6-1

$$\begin{aligned}
S_0(E) &= \frac{\partial\epsilon(E)}{\partial\epsilon_0} = e^{-\frac{E}{E_0}},\\
S_E(E) &= \frac{\partial\epsilon(E)}{\partial E_0} = \frac{\epsilon_0 E}{E_0^2}e^{-\frac{E}{E_0}},\\
S_c(E) &= \frac{\partial\epsilon(E)}{\partial\epsilon_c} = 1.
\end{aligned}$$

6-2

$$
\begin{aligned}
\mathrm{Cov}\left(\epsilon(E_i),\epsilon(E_j)\right) &= S_0(E_i)S_0(E_j)\mathrm{Cov}(\epsilon_0,\epsilon_0)+S_0(E_i)S_E(E_j)\mathrm{Cov}(\epsilon_0,E_0)+S_0(E_i)S_c(E_j)\mathrm{Cov}(\epsilon_0,\epsilon_c)\\
&+ S_E(E_i)S_0(E_j)\mathrm{Cov}(E_0,\epsilon_0)+S_E(E_i)S_E(E_j)\mathrm{Cov}(E_0,E_0)+S_E(E_i)S_c(E_j)\mathrm{Cov}(E_0,\epsilon_c)\\
&+ S_c(E_i)S_0(E_j)\mathrm{Cov}(\epsilon_c,\epsilon_0)+S_c(E_i)S_E(E_j)\mathrm{Cov}(\epsilon_c,E_c)+S_c(E_i)S_c(E_j)\mathrm{Cov}(\epsilon_c,\epsilon_c)\\
&= S_0(E_i)S_0(E_j)\mathrm{Cov}(\epsilon_0,\epsilon_0)+S_E(E_i)S_E(E_j)\mathrm{Cov}(E_0,E_0)+S_c(E_i)S_c(E_j)\mathrm{Cov}(\epsilon_c,\epsilon_c)\\
&+ \left[S_0(E_i)S_E(E_j)+S_0(E_j)S_E(E_i)\right]\mathrm{Cov}(\epsilon_0,E_0)\\
&+ \left[S_0(E_i)S_c(E_j)+S_0(E_j)S_c(E_i)\right]\mathrm{Cov}(\epsilon_0,\epsilon_c)\\
&+ \left[S_E(E_i)S_c(E_j)+S_E(E_j)S_c(E_i)\right]\mathrm{Cov}(E_0,\epsilon_c)\\
&= S_0(E_i)S_0(E_j)(\Delta\epsilon_0)^2+S_E(E_i)S_E(E_j)(\Delta E_0)^2+(\Delta\epsilon_c)^2\\
&+ \left[S_0(E_i)S_E(E_j)+S_0(E_j)S_E(E_i)\right]\mathrm{Cov}(\epsilon_0,E_0)\\
&+ \left[S_0(E_i)+S_0(E_j)\right]\mathrm{Cov}(\epsilon_0,\epsilon_c)\\
&+ \left[S_E(E_i)+S_E(E_j)\right]\mathrm{Cov}(E_0,\epsilon_c)\\
&= e^{-\frac{E_i+E_j}{E_0}}(\Delta\epsilon_0)^2+\frac{\epsilon_0^2E_iE_j}{E_0^4}e^{-\frac{E_i+E_j}{E_0}}(\Delta E_0)^2+(\Delta\epsilon_c)^2\\
&+ \epsilon_0\frac{E_i+E_j}{E_0^2}e^{-\frac{E_i+E_j}{E_0}}\mathrm{Cov}(\epsilon_0,E_0)\\
&+ \left(e^{-\frac{E_i}{E_0}}+e^{-\frac{E_j}{E_0}}\right)\mathrm{Cov}(\epsilon_0,\epsilon_c)\\
&+ \frac{\epsilon_0}{E_0^2}\left(E_ie^{-\frac{E_i}{E_0}}+E_je^{-\frac{E_j}{E_0}}\right)\mathrm{Cov}(E_0,\epsilon_c).
\end{aligned}
$$

6-3

$$
\begin{aligned}
(\Delta\epsilon_0)^2 &= 0.01,\\
(\Delta E_0)^2 &= 100\ (\mathrm{keV}^2),\\
(\Delta\epsilon_c)^2 &= 0.0001,\\
\mathrm{Cov}(\epsilon_0,E_0) &= \Delta\epsilon_0\Delta E_0\mathrm{Cor}(\epsilon_0,E_0)=-0.8\ (\mathrm{keV}),\\
\mathrm{Cov}(\epsilon_0,\epsilon_c) &= \Delta\epsilon_0\Delta\epsilon_c\mathrm{Cor}(\epsilon_0,\epsilon_c)=0.0004,\\
\mathrm{Cov}(E_0,\epsilon_c) &= \Delta E_0\Delta\epsilon_c\mathrm{Cor}(E_0,\epsilon_c)=-0.07\ (\mathrm{keV}).
\end{aligned}
$$

6-4 First, the equation to obtain the variance in the interpolated detection efficiency is

$$
\begin{aligned}
\mathrm{Var}(\epsilon(E)) &= \mathrm{Cov}(\epsilon(E),\epsilon(E))\\
&= e^{-\frac{2E}{E_0}}(\Delta\epsilon_0)^2+\frac{\epsilon_0^2E^2}{E_0^4}e^{-\frac{2E}{E_0}}(\Delta E_0)^2+(\Delta\epsilon_c)^2\\
&+ \frac{2\epsilon_0E}{E_0^2}e^{-\frac{2E}{E_0}}\mathrm{Cov}(\epsilon_0,E_0)\\
&+ 2e^{-\frac{E}{E_0}}\mathrm{Cov}(\epsilon_0,\epsilon_c)\\
&+ \frac{2\epsilon_0E}{E_0^2}e^{-\frac{E}{E_0}}\mathrm{Cov}(E_0,\epsilon_c).
\end{aligned}
$$

The interpolated efficiencies at two energies and their uncertainties are

$$\epsilon_{800} \sim 0.6779 \pm 0.0139,$$
$$\epsilon_{1000} \sim 0.5427 \pm 0.0090,$$
$$\mathrm{Cov}(\epsilon_{800}, \epsilon_{1000}) \sim 0.0001160.$$

6-5

$$\begin{aligned}
\Delta\epsilon_{800}/\epsilon_{800} &\sim 0.0139/0.6779 \sim 2.053\%, \\
\Delta\epsilon_{1000}/\epsilon_{1000} &\sim 0.0090/0.5427 \sim 1.650\%. \\
\mathrm{cov}(\epsilon_{800}, \epsilon_{1000}) &\sim 0.0001160/(0.6779 \cdot 0.5427) \sim 3.153\%^2.
\end{aligned}$$

6-6 If we set $\epsilon_i = \epsilon(E_i)$ and $\epsilon_j = \epsilon(E_j)$, we can write $\eta = \eta(\epsilon_i, \epsilon_j) = \epsilon_i/\epsilon_j$. Then

$$\begin{aligned}
\mathrm{var}(\eta) &= \left(\frac{\epsilon_i}{\eta}\frac{\partial\eta}{\partial\epsilon_i}\right)\mathrm{cov}(\epsilon_i, \epsilon_i)\left(\frac{\epsilon_i}{\eta}\frac{\partial\eta}{\partial\epsilon_i}\right) + \left(\frac{\epsilon_i}{\eta}\frac{\partial\eta}{\partial\epsilon_i}\right)\mathrm{cov}(\epsilon_i, \epsilon_j)\left(\frac{\epsilon_j}{\eta}\frac{\partial\eta}{\partial\epsilon_j}\right) \\
&+ \left(\frac{\epsilon_j}{\eta}\frac{\partial\eta}{\partial\epsilon_j}\right)\mathrm{cov}(\epsilon_j, \epsilon_i)\left(\frac{\epsilon_i}{\eta}\frac{\partial\eta}{\partial\epsilon_i}\right) + \left(\frac{\epsilon_j}{\eta}\frac{\partial\eta}{\partial\epsilon_j}\right)\mathrm{cov}(\epsilon_j, \epsilon_j)\left(\frac{\epsilon_j}{\eta}\frac{\partial\eta}{\partial\epsilon_j}\right) \\
&= \mathrm{cov}(\epsilon_i, \epsilon_i) - \mathrm{cov}(\epsilon_i, \epsilon_j) - \mathrm{cov}(\epsilon_j, \epsilon_i) + \mathrm{cov}(\epsilon_j, \epsilon_j) \\
&= \mathrm{var}(\epsilon_i) + \mathrm{var}(\epsilon_j) - 2\mathrm{cov}(\epsilon_i, \epsilon_j).
\end{aligned}$$

6-7

$$\begin{aligned}
\eta_{800,1000} &= \epsilon_{800}/\epsilon_{1000} \sim 1.2492. \\
\mathrm{var}(\eta_{800,1000}) &= \mathrm{var}(\epsilon_{800}) + \mathrm{var}(\epsilon_{1000}) - 2\mathrm{cov}(\epsilon_{800}, \epsilon_{1000}) \\
&\sim (2.053\%)^2 + (1.650\%)^2 - 2 \times 3.153\%^2 \sim 0.6312\%^2. \\
\Delta\eta_{800,1000}/\eta_{800,1000} &= \sqrt{\mathrm{var}(\eta_{800,1000})} \sim \sqrt{0.6312\%^2} \sim 0.7945\%.
\end{aligned}$$

The fractional uncertainty of the detection efficiency ratio (~0.8%) is smaller than the fractional uncertainty of the detection efficiencies (~2%) because of the covariance term.