International Atomic Energy Agency

Random Variable

- mean, variance, standard variation, uncertainty -

Naohiko Otsuka

IAEA Nuclear Data Section

Stochastic Nature of Measurement

We believe physical quantity has a unique true value (within classical approx.) which is what evaluators want to determine.

Experimental results are stochastic due to statistical fluctuation and limitations of the measurement procedures – *random variables*.

Example: Number of events *N*

By repeating a counting experiment n times, we obtain a set of counting number

$\{N\}=N_1, N_2,\ldots,N_n$

They do not agree in general. N is a random variable.

Probability Distribution

We believe that a random variable distributes with a peak around the true value (probability distribution).

In general, this distribution is described by

P_k : probability for a <u>discrete</u> random variable k.

$P(x)$: probability (density) for a <u>continuous</u> random variable x.

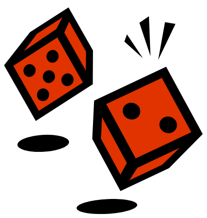
By definition,

$\Sigma_k P_k = 1$ (discrete random variable)

$\int dx\, P(x)=1$ (continuous random variable)

Discrete Random Variables – One Dice

Probability to find a value (1, 2, 3, 4, 5 or 6) on a dice

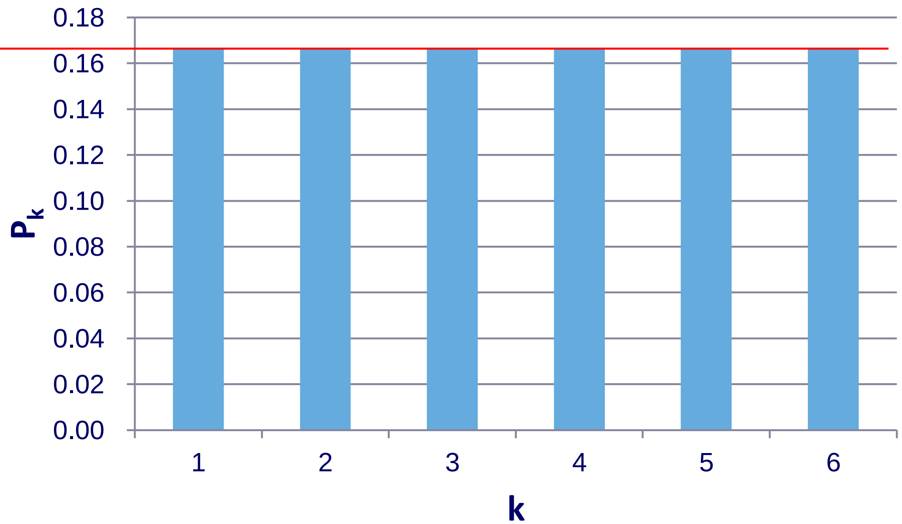


k (=1,2,, or 6): random variable

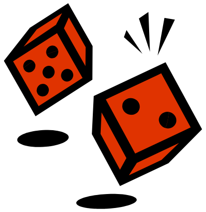
P_k (= 1/6 for each k): probability distribution

k	1	2	3	4	5	6
P_k	1/6	1/6	1/6	1/6	1/6	1/6

1/6 →



Discrete Random Variables: Sum from Two Dices



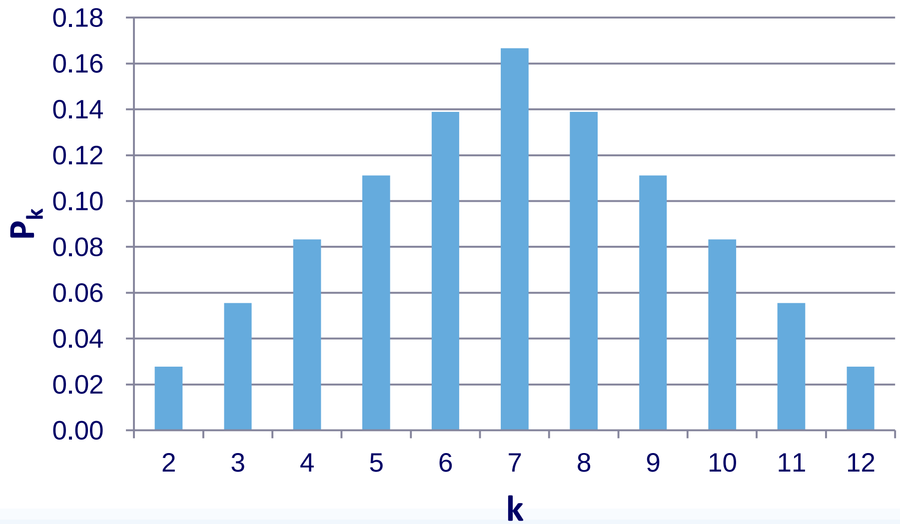
Probability to find k=i+j with i and j (=1,,6) on two dices

i=(1,2,, or 6) – random variable

j=(1,2,, or 6) – random variable

→ k=(2,3,4,…,12) is also a random variables.

k	2	3	4	5	6	7	8	9	10	11	12
P_k	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36



Mean, Variance and Standard Deviation (Discrete Variable)

Definition of mean, variance and standard deviation for a discrete random variable k following the probability distribution P_k:

- **Mean**

  $<k> = \Sigma_{k=1,n} k\cdot P_k$

- **Variance**

  $v = <(k-<k>)^2> = <k^2>-<k>^2$

- **Standard deviation**

  $\Delta k=(v)^{1/2}$

Mean and standard deviation are often adopted as "best estimate" and "uncertainty" (will be discussed later).

Mean, Variance and Standard Deviation (Continuous Variable)

For a continuous random variable x, similarly

- **Mean**

  $<x> = \int dx\, x \cdot P(x)$

- **Variance**

  $v = <(x-<x>)^2> = <x^2>-<x>^2$

- **Standard deviation**

  $\Delta x=(v)^{1/2}$

Population and Sample

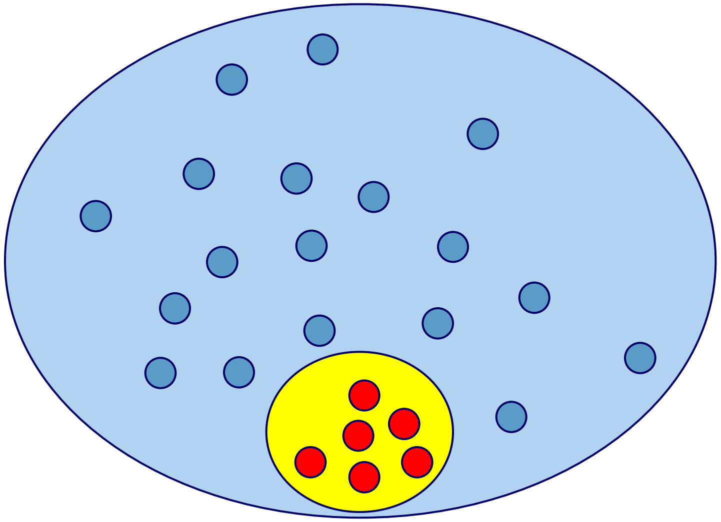
In general, we cannot know the probability distribution (e.g., 1/6) without experiment.

We cannot measure the whole set ("population") to extract the statistical property.

The statistical property of the population may be estimated by <u>sampling</u> (size n).

For a discrete random variable k,

- mean $\langle k \rangle' = \Sigma_{i=1,n} k_i / n$
- variance $v' = \Sigma_{i=1,n} k_i^2 / n - \langle k \rangle'^2$
- standard deviation $\Delta k' = (v')^{1/2}$



Population and Sample (cont)

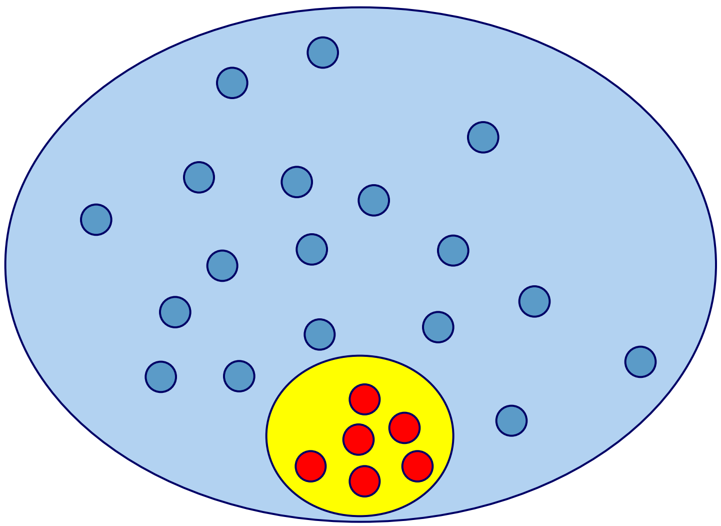
The properties of the population is related with those of a size n sample ($\langle k \rangle'$, v', $\Delta k'$) as follows:

- $\langle k \rangle = \langle k \rangle'$ (equal!)
- $v = v'$ $[n/(n-1)]$
- $\Delta k = \Delta k' [(n-1)/2]^{1/2} \Gamma[(n-1)/2] / \Gamma(n/2)$

Γ: gamma function

If the sample size n is large enough, n-dependent factors ~ 1.

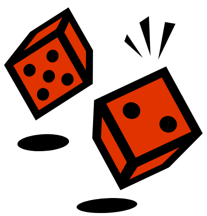
Population ($\langle k \rangle$, v, Δk)



Sample (n samples)
($\langle k \rangle'$, v', $\Delta k'$)

Mean, Variance and Standard Deviation: One Dice

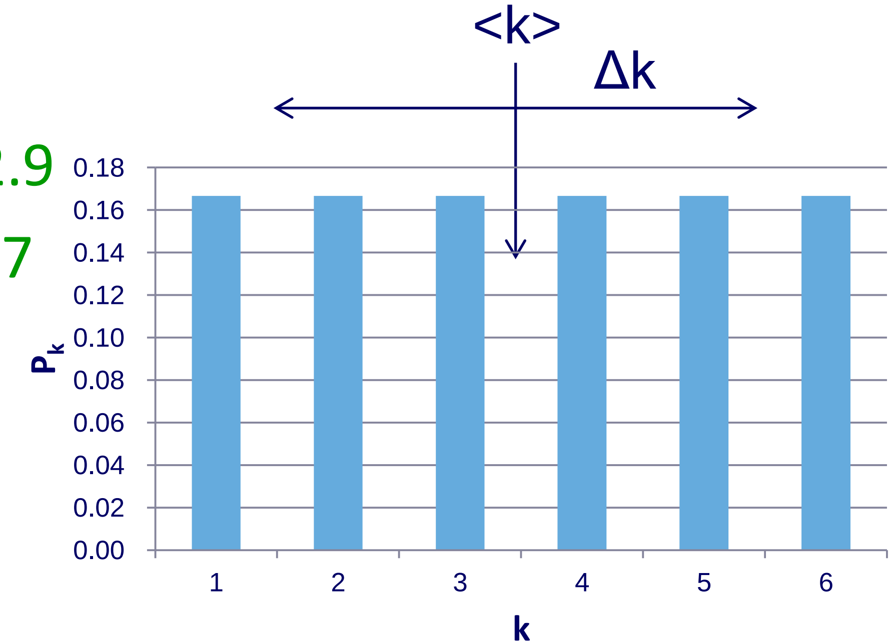
k	1	2	3	4	5	6
P_k	1/6	1/6	1/6	1/6	1/6	1/6



k: number on a dice

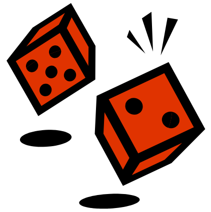
- mean $\langle k\rangle = \Sigma_{k=1,6}\, k\cdot P_k \sim 3.5$
- variance $v = \Sigma_{k=1,6}\, k^2\cdot P_k - \langle k\rangle^2 \sim 2.9$
- standard deviation $\Delta k=(v)^{1/2} \sim 1.7$

Even if the probability is equally distributed, we can define mean and standard deviation.



There is *no* concept of “true value” and “uncertainty”.

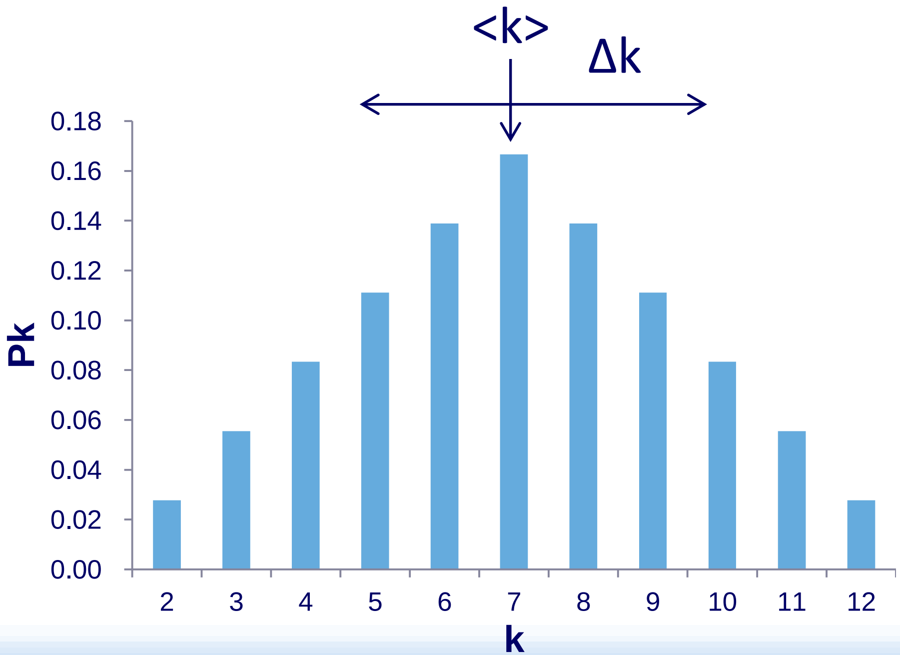
Mean, Variance and Standard Deviation: Two Dices



k=m+n with m and n (m, n=1,,6) on two dices

k	2	3	4	5	6	7	8	9	10	11	12
P_k	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

- mean $\langle k \rangle = \Sigma_k\, k \cdot P_k \sim 7$
- variance $v = \Sigma_k\, k^2 \cdot P_k - \langle k \rangle^2 \sim 5.8$
- standard deviation $\Delta k = (v_k)^{1/2} \sim 2.4$



Poisson Distribution

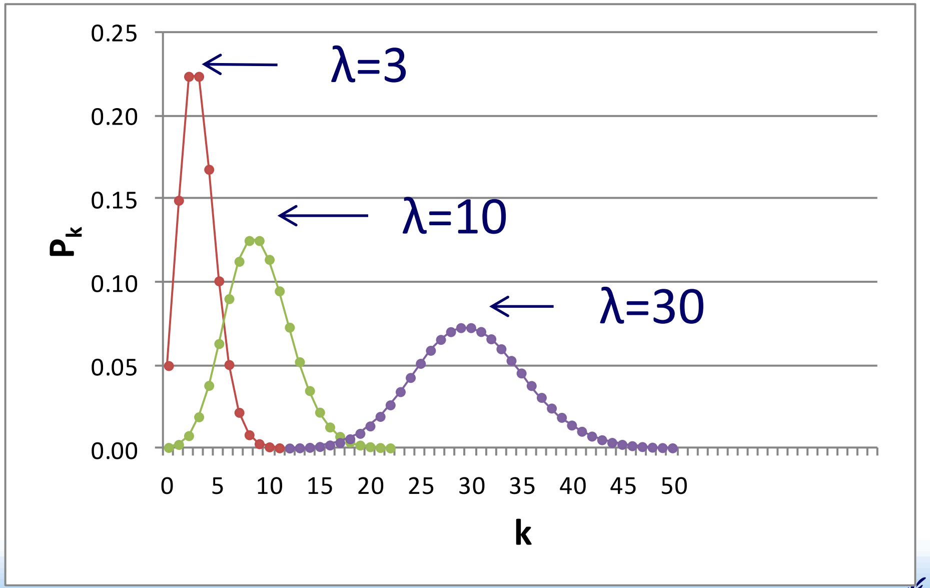
If the event occur

- with a known mean rate (= λ events in a given time span ΔT);
- independently of the time since the last event;
- one time at maximum within an appropriate Δt (<< ΔT);

the probability distribution is described by **Poisson distribution**:

$P_k = \lambda^k \exp(-\lambda)/k!$

Siméon Denis Poisson
(1781 – 1840)



Occurs Independently of the Time Since the Last Event?

20	- - - - - - - - - 54
21	00 07 15 22 30 37 45 52
22	00 07 15 22 30 37 45 52
23	00 07 15 22 30 37 45 52
0	00 07 15 23 35

Occurs dependently of the time elapsed since the last subway

Basic Properties of Poisson Distribution

You may prove that Poisson distribution $P_k = \lambda^k \exp(-\lambda)/k!$ satisfies

- normalization $\Sigma_{k=0,\infty} P_k = 1$
- mean $\langle k \rangle = \Sigma_{k=0,\infty} k \cdot P_k = \lambda$
- variance $v = \Sigma_{k=0,\infty} k^2 \cdot P_k - \langle k \rangle^2 = \lambda$
- standard deviation $\Delta k = (v)^{1/2} = \lambda^{1/2}$

mean = variance!

Poisson Distribution: Counting Experiment

Suppose we repeated counting experiment n times, and obtain count N_i (i=1,,n). If the phenomenon follows the Poisson distribution, we obtain

- mean $\langle N \rangle = (\Sigma_{i=1,n} N_i)/n$
- variance $v = (\Sigma_{i=1,n} N_i^2) / n - \langle N \rangle^2 \sim \langle N \rangle$
- standard deviation $\Delta N = v^{1/2} \sim \langle N \rangle^{1/2}$

Counting Statistics and Irradiation Time

Measurement of an yield Y=N/ε by measuring count N with a detector (efficiency ε, Δε/ε=5.0%). We expect 500 counts/min.

N	ΔN/N (%)	Δε/ε (%)	ΔY/Y (%)	time (min)
100	10.0	5.0	11.2	0.2
500	4.5	5.0	6.7	1.0
1,000	3.2	5.0	5.9	2.0
10,000	1.4	5.0	5.2	20.0
20,000	1.0	5.0	5.1	40.0
50,000	0.4	5.0	5.0	100.0
100,000	0.3	5.0	5.0	200.0
200,000	0.2	5.0	5.0	333.3

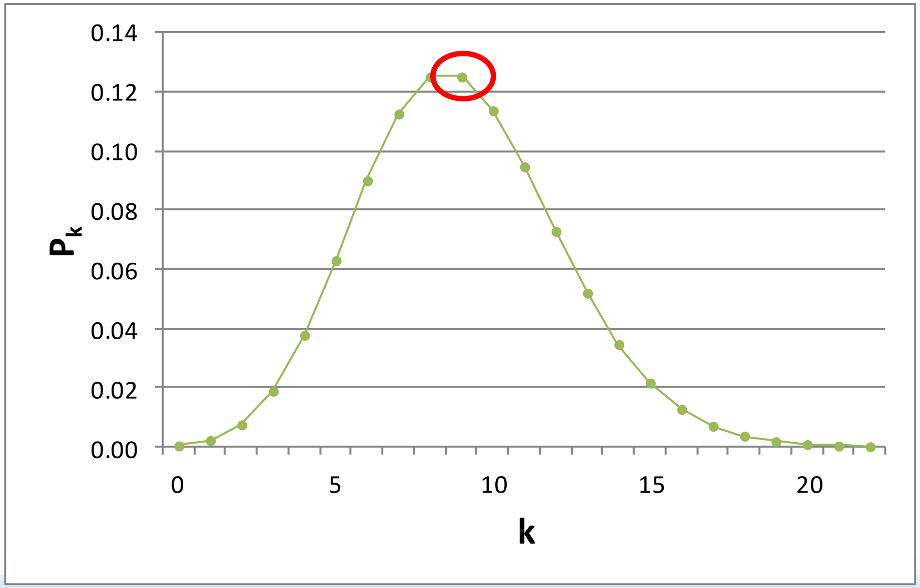
Counting more than ~100 min does not improve the uncertainty!

Mean and Standard Deviation from One Measurement

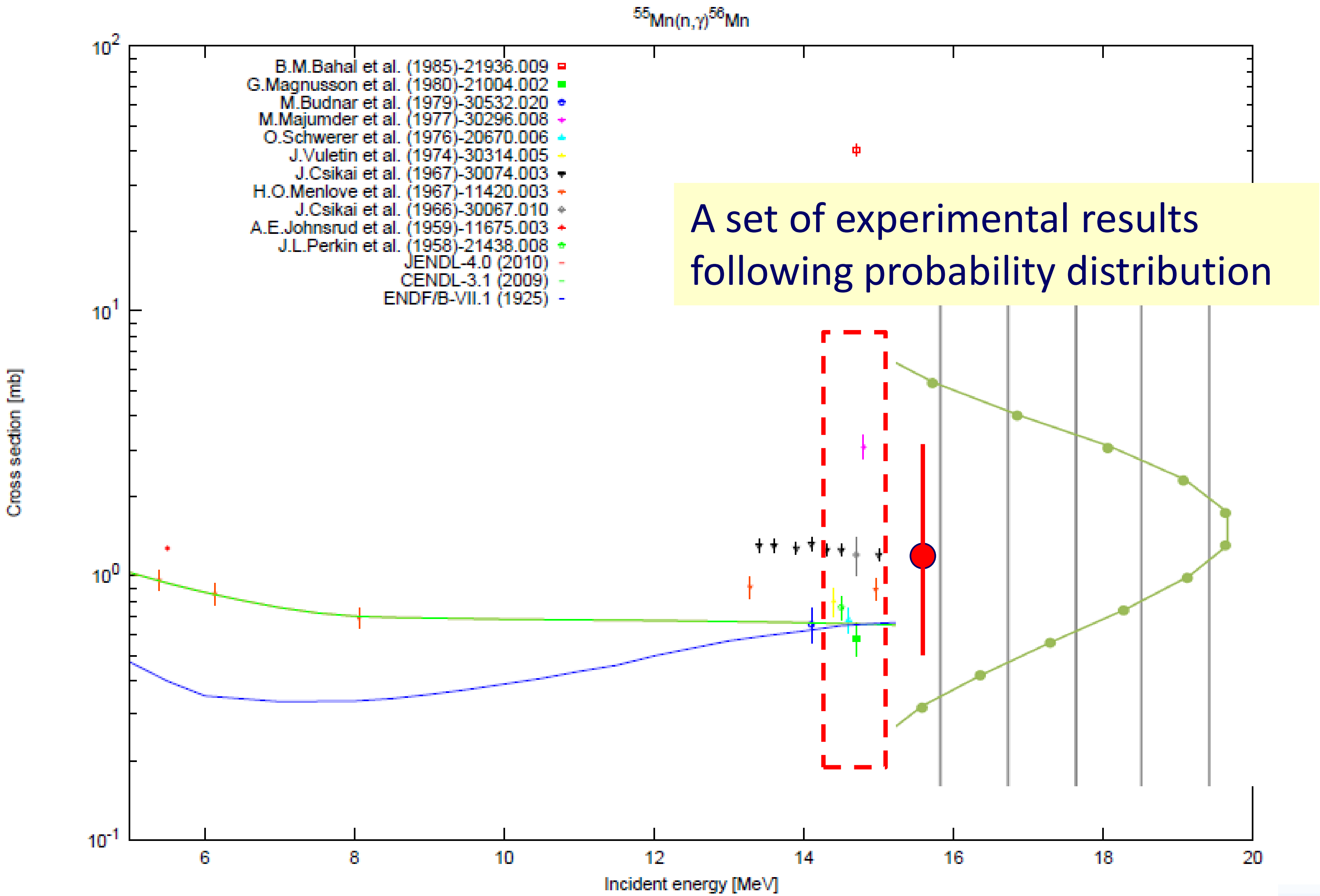
In nuclear reaction measurements, count N from a single counting measurement (namely i=1) is treated as <N>. (Namely $N \sim \langle N \rangle$, $\Delta N \sim \langle N \rangle^{1/2}$).

Example:

A single measurement gives $N_1 = \langle N \rangle = 10$ without repeating the experiment.



Cross Section Evaluation

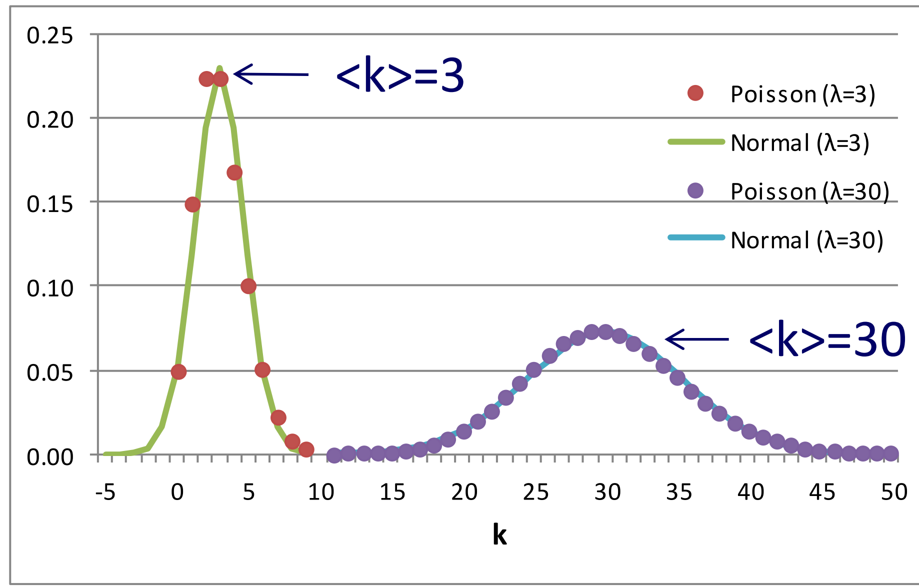


Normal (Gauss) Distribution

For an enough large mean number, the Poisson distribution is well approximated by the **normal (Gauss) distribution**

$P_k = \lambda^k \exp(-\lambda)/k! \rightarrow P(k) = \exp[-(k-\lambda)^2/(2\lambda)] / (2\pi\lambda)^{1/2}$

Johann Carl Friedrich Gauß (1777 – 1855)



Note that P(k) is a probability density distribution. The probability to find a value k within $[x_{min},x_{max}]$ is $P_k \sim \int_{xmin}^{xmax} dx\, P(x)$.

Properties of Normal (Gauss) Distribution

You may prove that normal distribution, $P(x) = \exp[-(x-\lambda)^2/(2\lambda)] / (2\pi\lambda)^{1/2}$ satisfies that

- Normalization $\int_{-\infty}^{+\infty} dx\, P(x) = 1$
- mean $x_0 = \langle x \rangle = \int_{-\infty}^{+\infty} dx\, x \cdot P(x) = \lambda$
- variance $v = \int_{-\infty}^{+\infty} dx\, x^2 \cdot P(x) - \langle x \rangle^2 = \lambda$
- standard deviation $\Delta x = (v)^{1/2} = \lambda^{1/2}$

$\int_{-\infty}^{+\infty} dx\, \exp(-ax^2) = (\pi/a)^{1/2}$,

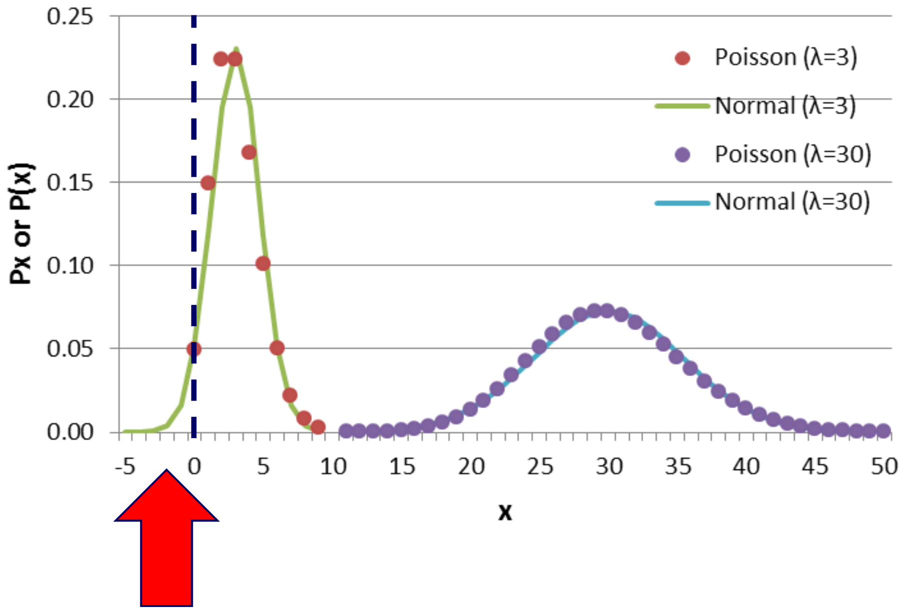
$\int_{-\infty}^{+\infty} dx\, x \cdot \exp(-ax^2) = 0$,

$\int_{-\infty}^{+\infty} dx\, x^2 \cdot \exp(-ax^2) = (\pi^{1/2})/(2a^{3/2})$

Remarks on Normal Distribution

Poisson distribution P_x is defined for a non-negative random variables x.

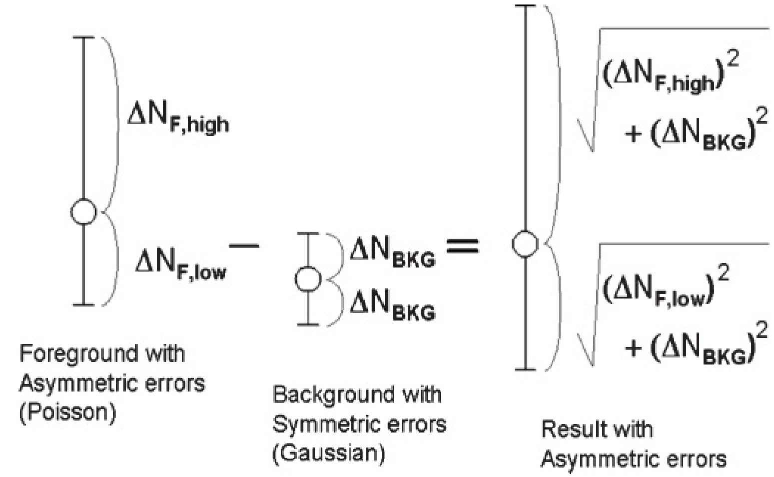
However normal distribution P(x) may give finite probability for negative x.

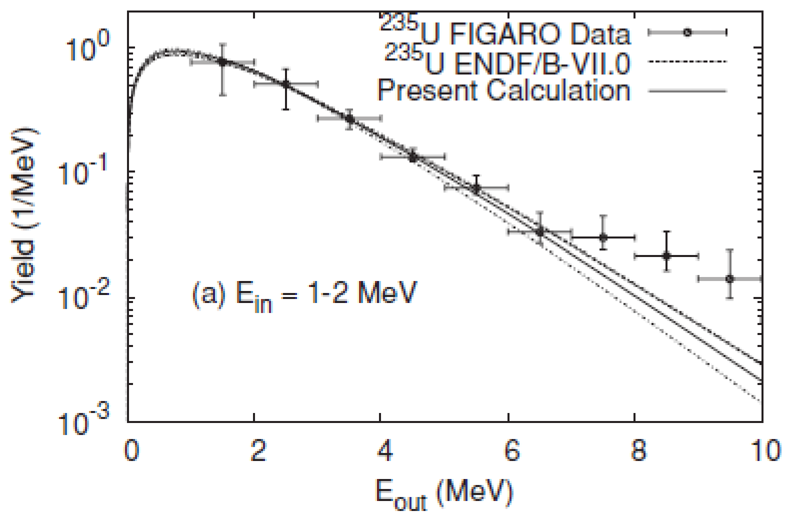


The normal distribution is a good approximation of the Poisson distribution when we have enough counting number x.

An Example of non-Gaussian Poisson Distribution

In our experiments, the foreground statistics are poor, and there are some zero counts in the energy-binned foreground spectra so that it may be inappropriate to assume Gaussian distributions for the data. The error bars of foreground events are estimated with ROOFIT [27] using the Poisson distribution, which is not symmetric. Here we describe the upper side of the error bar, $\Delta N_{\mathrm{F,\,high}}$, and the lower side, $\Delta N_{\mathrm{F,\,low}}$.



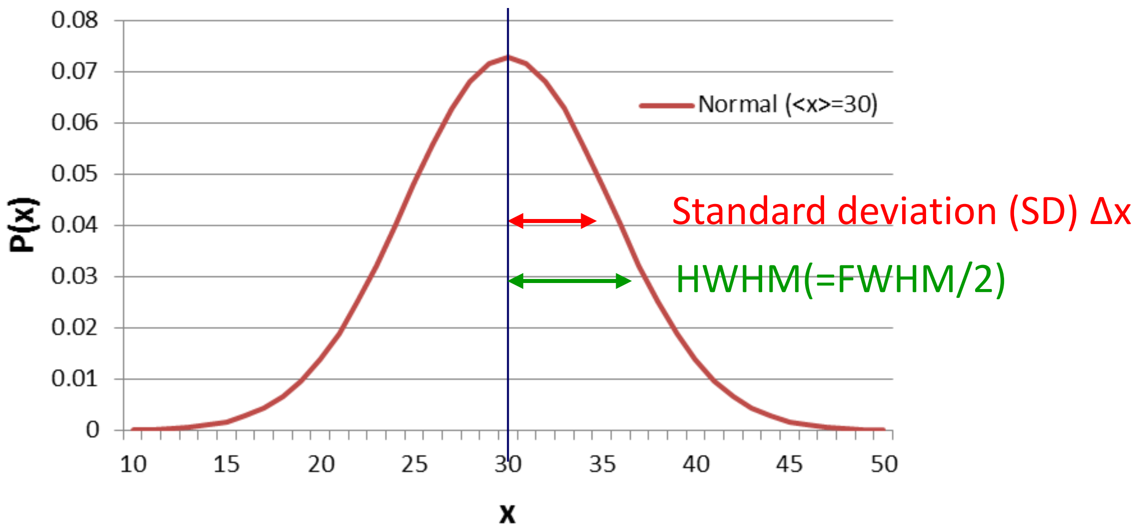



```
SUBENT        14290002   20111006
BIB                  3          7
REACTION    (92-U-235(N,F),PR,NU/DE,,AV/REL)
             Spectra are normalized to unity by integrating the
              spectra in the 2.0-6.5 MeV emission energy range
ERR-ANALYS  (ERR-S) Statistical uncertainty, considering
                    - foregound counting statistics  (Poisson)
                    - background counting statistics (Gaussian)
STATUS      (TABLE) Plotted in Fig.7 of Phys.Rev.C83(2011)034604
ENDBIB               7
NOCOMMON             0          0
DATA                 6         63
EN-MIN      EN-MAX      E          DATA       -ERR-S     +ERR-S
MEV         MEV         MEV        ARB-UNITS  ARB-UNITS  ARB-UNITS
 1.          2.            1.5      7.6634E-01 3.4578E-01 2.9948E-01
 1.          2.            2.5      5.1220E-01 1.9031E-01 1.6367E-01
 1.          2.            3.5      2.7136E-01 4.7978E-02 4.5734E-02
```

S. Noda et al., Phys. Rev. C 83(2011)034604 (EXFOR 14290)

Normal Distribution:Standard Deviation (SD) and HWHM

Each result falls within <x>± SD in 68% probability (confidential level) in the normal distribution (SD: standard deviation s).



By using the definition of the normal distribution, you can easily prove that (HWHM) = $(2 \ln 2)\ \Delta x \sim 1.2\ \Delta x$.

Uncertainty ≠ Resolution (Example)

Both *uncertainty* and *resolution* are often related with the Gaussian shape.

However they are different and must be distinguished:

Uncertainty:

Statistical fluctuation around the true (mean) value of a quantity. It becomes smaller if we repeat the measurement.

Resolution (Dispersion):

"True" distribution. It does *not* become smaller even if we repeat the measurement.

Example: Fission Fragment Mass Distribution

Table 3
Global characteristics of the fission fragments' mass and energy distributions of ^{236,238,240,242,244}Pu(SF). The indicated errors are only statistical

	N	$\langle E_k^* \rangle$ (MeV)	$\sigma_{E_k^*}$ (MeV)	$\langle M_H^* \rangle$ (amu)	$\sigma_{M_H^*}$ (amu)
^{236}Pu(SF)	1977	175.3±0.3	11.0 ±0.2	139.1 ±0.1	5.3 ±0.1
^{238}Pu(SF)	2051	176.4±0.3	11.3 ±0.2	139.4 ±0.1	5.9 ±0.1
^{240}Pu(SF)	11867	178.5±0.1	11.5 ±0.1	138.87±0.05	5.76±0.04
^{242}Pu(SF)	31722	180.5±0.1	11.52±0.04	137.89±0.03	5.24±0.02
^{244}Pu(SF)	17541	179.0±0.1	11.1 ±0.1	138.32±0.04	5.77±0.03

Mean mass and its uncertainty

Mass distribution width (standard deviation) and its uncertainty

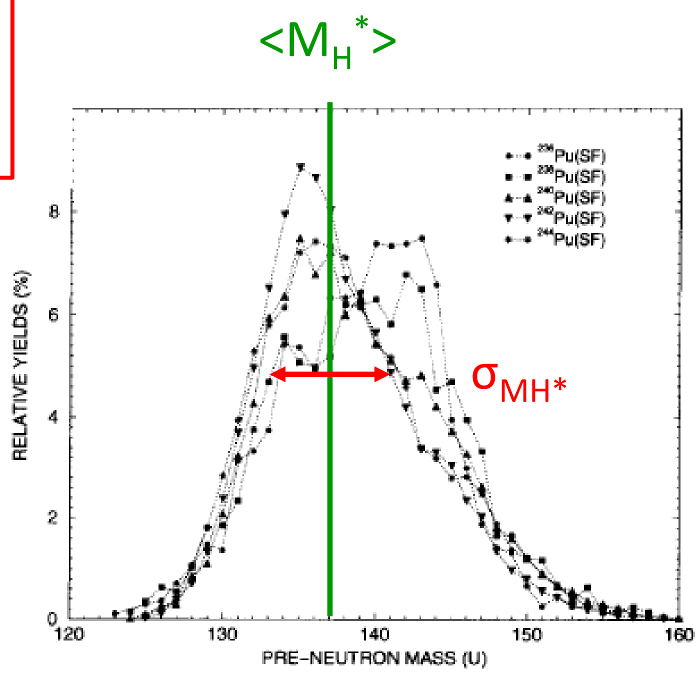


Fig. 3. Pre-neutron heavy fragment mass distributions of ^{236,238,240,242,244}Pu(SF).

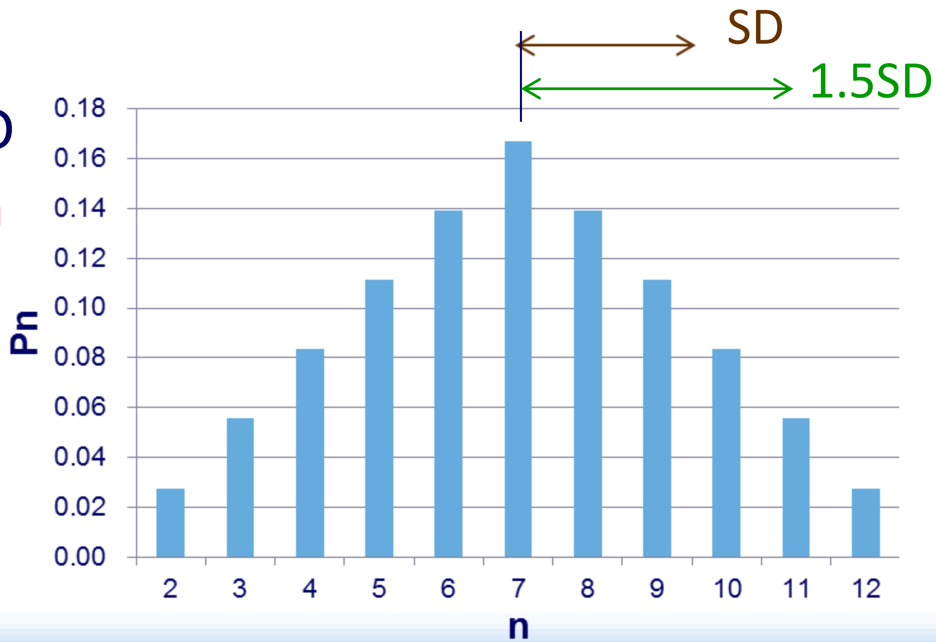
L. Dematte et al. Nucl.Phys.A617(1997)331

Best Estimate and Uncertainty

The mean and standard deviation (SD) are often adopted as the best estimate and uncertainty, respectively.

The definition of "uncertainty" is *not unique*.

For example, one may adopt 1.5SD instead of SD as another definition of the uncertainty.



Definition of Uncertainty
("Review of Particle Physics")

1. PHYSICAL CONSTANTS

Table 1.1. Reviewed 2011 by P.J. Mohr (NIST). Mainly from the "CODATA Recommended Values of the F 2010" by P.J. Mohr, B.N. Taylor, and D.B. Newell in `arXiv:1203.5425` and Rev. Mod. Phys. (to be publish (beginning with the Fermi coupling constant) comes from the Particle Data Group. The figures in parentheses after the values give the 1-standard-deviation uncertainties in the last digits; the corresponding fractional uncertainties in parts per 10^9 (ppb) are given in the last column. This set of constants (aside from the last group) is recommended for international use by CODATA (the Committee on Data for Science and Technology). The full 2010 CODATA set of constants may be found at `http://physics.nist.gov/constants`. See also P.J. Mohr and D.B. Newell, "Resource Letter FC-1: The Physics of Fundamental Constants," Am. J. Phys, **78** (2010) 338.

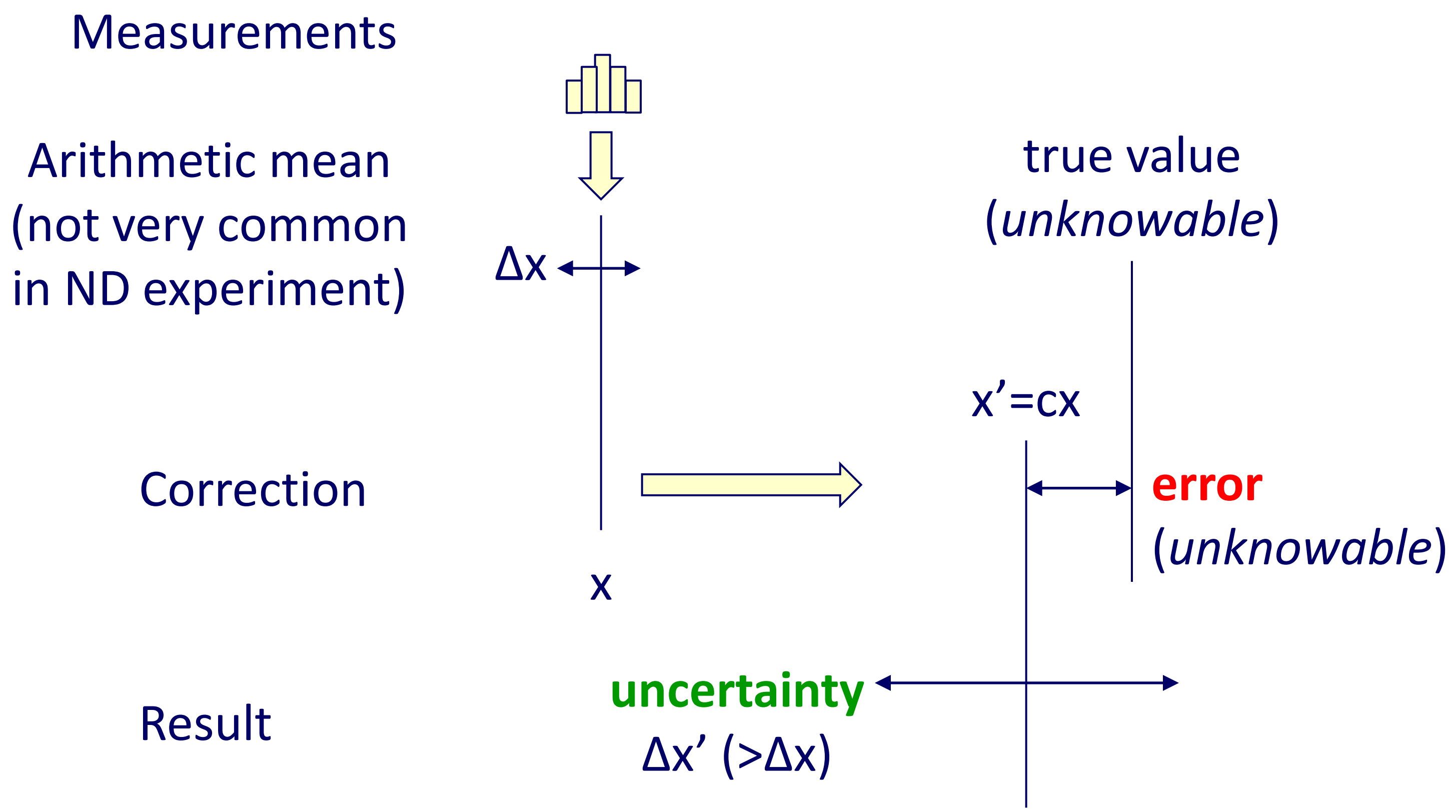
Quantity	Symbol, equation	Value	Uncertainty (ppb)
speed of light in vacuum	c	$299\,792\,458$ m s^{-1}	exact*
Planck constant	h	$6.626\,069\,57(29)\times10^{-34}$ J s	44
Planck constant, reduced	$\hbar \equiv h/2\pi$	$1.054\,571\,726(47)\times10^{-34}$ J s	44
		$= 6.582\,119\,28(15)\times10^{-22}$ MeV s	22
electron charge magnitude	e	$1.602\,176\,565(35)\times10^{-19}$ C $= 4.803\,204\,50(11)\times10^{-10}$ esu	22, 22
conversion constant	$\hbar c$	$197.326\,9718(44)$ MeV fm	22
conversion constant	$(\hbar c)^2$	$0.389\,379\,338(17)$ GeV^2 mbarn	44
electron mass	m_e	$0.510\,998\,928(11)$ MeV/c^2 $= 9.109\,382\,91(40)\times10^{-31}$ kg	22, 44

J. Beringer *et al.* (Particle Data Group), Phys. Rev. D**86**, 010001 (2012)

The caption clearly states that 1SD is adopted as the uncertainty in the table.

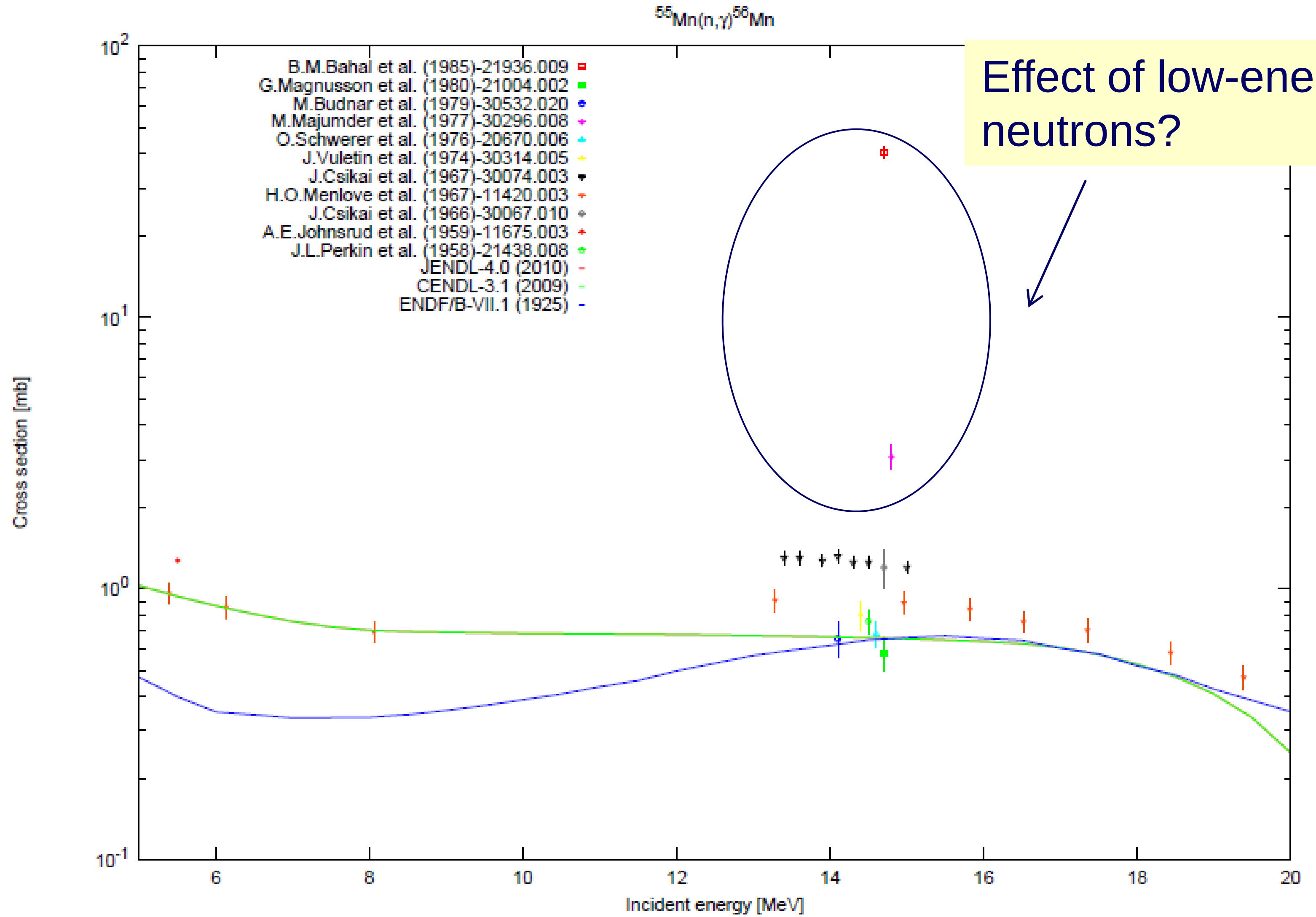
Uncertainty and Error

c.f. Fig.D.2 of GUM2008



The true value is within the uncertainty.
(successful estimation)

Example of Corrections– $^{55}Mn(n,\gamma)^{56}Mn$



Effect of low-energy neutrons?

Various Corrections for Lower Energy Neutrons

TABLE 1

Essential experimental data and principal features of the decay schemes used for the evaluation of the activation cross sections

Product nucleus	Sample: material [a]	Sample: purity [b] (%)	Sample: weight (mg)	Decay scheme: γ-transition used for evaluation: energy (keV)	Decay scheme: γ-transition used for evaluation: int. per 100 decays	Decay scheme: half-life	Decay scheme: ref.	Corrections [c]: lower energy neutrons (% of measured activity): thermal, epithermal	Corrections [c]: lower energy neutrons (% of measured activity): prod. in target	Corrections [c]: lower energy neutrons (% of measured activity): prod. in sample	Corrections [c]: γ-absorption in sample (%)
^{38}Cl	$BaCl_2$ p	99.998	658.8	2167	42	37.3 min	47)		1.8		
^{42}K	KI p	sp	594.5	1524	18.3	12.4 h	47)		not calculated		
^{51}Ti	Ti, sheet	99.97	115.64	320	95	5.76 min	48)		1.7		
^{52}V	V_2O_5 p	99	418.25	1134	100	3.75 min	48)		2.4	1.5	
^{56}Mn	Mn_3O_4 p	99.995	472.25	846.5	98.8	2.582 h	49)	0.5	5	0.5	1
^{72}Ga	Ga_2O_3 p	99.999	506.65	834.7	95	14.1 h	50)		13.5	1	1
^{88}Rb	RbCl p	sp	509.8	1836	23.2	17.8 min	51)		not calculated		
^{90}Y	Y_2O_3 p	99.9999	364.43	480	99.6	3.19 h	15)		1.5		1
^{128}I	KI p	sp	594.5	442.9	15.8	25 min	52, 53)	1	36	8.3	2
^{131g}Te	TeO_2 p	99.999	272.35	149.7	80	25 min	54–56)		6.9	2.0	4
^{131m}Te	TeO_2 p	99.999	272.35	334.5	14.4	1.25 d	54–56)		not calculated		1
^{139}Ba	$BaCl_2$ p	99.998	658.8	166	22.1	85 min	57, 51, 53)		2.8		5.7
^{140}La	La_2O_3 p	99.999	440.5	1596	95.3	40.2 h	51, 58)		7.3	0.4	
^{143}Ce	CeO_2 p	99.9	495.5	293.3	49.5	33.7 h	59)				2.3
^{187}W	WO_3 p	99.9	591	685.7	28.9	23.8 h	60)		16.5	5.2	1
^{199}Pt	Pt, sheet	99.97	2161.2	542.7	15.5	31 min	61, 53)		27	30	8.2
^{198}Au	Au, sheet	99.99	198.75	412	99.82	2.698 d	62)	3.35	31.9	3.8	2

[a]) The abbreviation p is used for powder.

[b]) The abbreviation sp is used for suprapure, according to definition by Merck Laboratories, Germany, who supplied target materials in these cases. In all other cases target materials were supplied by Koch-Light Laboratories Ltd., England, or by Goodfellow Metals Ltd., England.

[c]) Where no value appears the correction was negligible.

O. Schwerer et al., Nucl. Phys. **A264**(1976)105 (EXFOR 20670)

Note: Correction procedures improve the best estimate, but also introduce a new source of uncertainty.

Summary

- Direct observables are random variables.
- Standard deviation (square root of variance) is often adopted as the "uncertainty".
- Poisson distribution: Mean=<N>, Uncertainty=<N>$^{1/2}$
- <N>=N, ΔN=<N>$^{1/2}$ are often done from a single measurement.
- If N is enough large, Poisson distribution →normal distribution.
- Uncertainty ≠Resolution, Uncertainty≠Error